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THE ACCURACY OF APPLYING LINEAR PISTON THEORY TO CYLINDRICAL SHELLS

by
Hans Krumhaar

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Graduate Aeronautical Laboratories
California Institute of Technology
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Abstract

In this report the aerodynamic pressure acting on a circular cylindrical shell of infinite length in an airflow parallel to the cylinder axis is studied. The shell is deformed by a harmonically oscillating standing wave of a sinusoidal pattern. Based upon the exact solutions, asymptotic expansions are developed for the aerodynamic pressure. In this manner the accuracy of the linear piston theory approximation, when applied to cylindrical shells, is investigated. Furthermore improved approximations can be obtained from these asymptotic expansions.

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List of Symbols

a_0	Velocity of sound in the undisturbed air.
$f(r)$	See eq. (1.4).
$H_n^{(1)}(z)$, $H_n^{(2)}(z)$	Hankel-functions (Bessel-functions of the third kind).
$K_n(z)$	Modified Hankel-functions (modified Bessel-functions of the third kind).
M	Mach number of the external airstream. M is positive if the air is blowing in the positive x-direction.
M_1	$M - \frac{\omega}{\nu a_0}$, see eq. (2.2).
M_2	$M + \frac{\omega}{\nu a_0}$, see eq. (2.5).
n	Number of waves around the circumference.
O	Landau symbol.
$\Delta p(x, r, \theta)$	Aerodynamic pressure disturbance due to the sinusoidal deformation of the cylinder, see eq. (1.6).
Δp	Aerodynamic pressure distrubance acting on the shell, due to the sinusoidal deformation or sinusoidal vibration of the cylindrical shell.
Δp^{*}	Ackeret's formula, see eq. (3.8).
Δp^{**}	Linear piston theory approximation, see eq. (4.2).
Δp^{***}	$a_0 \rho_0 \{a_0 M \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} - \frac{a_0}{2R} w\}$, see eq. (6.1).
R	Radius of the shell.

r	Radial coordinate.
$s(x)$	$\sin(\nu x)$ or $\cos(\nu x)$, see eq. (2.4b).
t	Time.
w	Deformation of the shell, positive in the positive r-direction.
w_0	Real or complex constant.
ϑ	See eq. (6.2).
x	Length coordinate.
x, r, θ	Cylinder coordinate system.
θ	Angular coordinate.
ν	Wave number, see eq. (1.4).
ξ	Real, positive variable.
ρ_0	Density of the undisturbed air.
ϕ	Velocity potential.
ω	Circular frequency of the shell vibration, see eq. (2.1), real and non-negative.

INTRODUCTION

Piston theory, originally developed by Lighthill (ref. 10), was introduced into aeroelasticity in the linearized form by Ashley and Zartarian as a handy tool in 1956, see ref. 2. This theory furnishes an approximation for the aerodynamic pressure acting on a slightly deformed flat plate in a supersonic airstream. The ~~linearized piston theory is widely used in the investigation of the~~ flutter of flat panels, see the summaries of Fung, refs. 5 and 6.

As far as the flutter of cylindrical shells is concerned, an appropriate approximation for the aerodynamic pressure acting on a vibrating shell is lacking. In the case of supersonic outer airstream, many authors use the linear piston theory expression for the aerodynamic pressure. Some authors even employ Ackeret's formula, neglecting the aerodynamic damping term of the linear piston theory.

There are doubts about the accuracy of using the linear piston theory for cylindrical shells. This opinion was strengthened by recent studies at Caltech. The author has investigated the flutter in an axisymmetrical mode of a thin cylindrical shell of finite length in a supersonic airstream on the bases of Timoshenko's linearized shell equations, linear piston theory and linear material damping, see ref. 7. On the other hand, wind tunnel experiments in the summers of 1961 and 1962 (see ref. 11) reveal no flutter in an axisymmetrical mode in the predicted Mach number range. This indicates that the basic physical assumptions on which the above

mentioned calculations are based are not satisfactory. The inadequacy of the linear piston theory and the neglect of boundary layer effects might be responsible at least partially for this disagreement.

The influence of the boundary layer, using a simplified model set forth by Fung in the summer of 1961, has been studied by Fung and Anderson, see refs. 1 and 6. It is the subject of the present paper to investigate the accuracy of the linear piston theory approximation when applied to cylindrical shells.

Leonhard and Hedgepeth develop in ref. 9 an exact expression for the aerodynamic pressure, acting on a cylindrical shell of infinite length, which is exposed externally to an airstream parallel to the generators of the cylinder, and where the shell is slightly deformed by a harmonically oscillating standing sinusoidal wave. In the present paper the well-known asymptotic expansions for cylinder functions are used to obtain an asymptotic formula for the aerodynamic pressure under consideration, in the form of polynomials of the reciprocal of the shell radius with a remainder term. The results differ according to whether the Mach number of the free stream or the frequency of the shell vibration is "large" or not.

In the case of "large" Mach numbers or "large" frequencies, the asymptotic expansion can be split up into the linear piston theory expression and some correction terms. The latter ones tend to zero if the radius of the shell and the Mach number tend to infinity; the same is true if the shell radius and the frequency of the shell vibration tend to infinity. Hence this expansion provides an

estimate of the error of the linear piston theory approximation. Furthermore improved approximations can be obtained from this asymptotic expansion.

In case neither the Mach number nor the frequency is "large", the linear piston theory approximation can no longer be considered as a first order approximation of the considered aerodynamic pressure.

Furthermore, on the assumption that the Mach number is larger than one and letting the shell radius tend to infinity and the frequency of the standing wave tend to zero, the asymptotic expansion tends to Ackeret's formula.

These investigations have been performed for standing waves of a sinusoidal pattern on the cylinder. The exact terms for the corresponding aerodynamic pressure, as well as most of the coefficients of the asymptotic expansion, depend also upon the frequency of the shell vibration and upon the wave length in the direction of the cylinder axis. Therefore, if these results are to be applied to more general vibration patterns, every term of the corresponding Fourier-expansion has to be treated specially.

It is with pleasure that the author expresses his gratitude to Professor Y. C. Fung for fruitful discussions and for bringing the above-mentioned paper of Leonhard and Hedgepeth again to the attention of the author. Furthermore the author wishes to express his thanks to Mrs. D. M. Eaton for her careful numerical calculations and to Mrs. E. Fox, who again typed the manuscript.

1. Aerodynamic Pressure Acting on a Stationary Sinusoidally Deformed Cylindrical Shell

We consider an infinitely long circular cylindrical shell of radius R, which is exposed externally to a uniform airstream parallel to the generators of the cylinder. The Mach number of the undisturbed airstream is M. The density and the velocity of sound of the air are denoted by ρ_0 and a_0 respectively. Let a cylinder coordinate system x, r, θ be chosen, where the positive direction of the x-axis coincides with the positive direction of the airstream[1]. Now the shell is assumed to be stationary and slightly deformed by a sinusoidal wave

$$w = w_0 \cos(n\theta)\, e^{-i\nu x} \tag{1.1}$$

where w is the radial displacement of the shell, measured positive in the outside direction. w_0 is a real or complex number, n is a non-negative integer and ν is a positive or negative number. The deformation (1.1) of the shell is assumed to be of infinitesimal amplitude, so that the differential equations and the boundary conditions for the air-flow are linearized, and the boundary conditions are applied on the mean position of the shell rather than on the actual surface. The governing equation for the velocity potential $\phi(x, r, \theta)$ reads (see e.g. ref. 4, page 432) for $|M| \neq 1$:

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(1) Therefore a negative Mach number means that the airstream is moving in the negative x-direction.

$$(1-M^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 . \qquad (1.2)$$

The boundary conditions are given by

$$\frac{\partial\phi}{\partial r} = a_0 M \frac{\partial w}{\partial x} \quad \text{for} \quad r = R; \qquad (1.3)$$

and Sommerfeld's finiteness and radiation condition for $r \to \infty$.

Solutions of eq. (1.2) which are suitable for our purposes are of the form

$$\phi(x,r,\theta) = f(r)\cos(n\theta)\, e^{-i\nu x}, \qquad (1.4)$$

where $f(r)$ is a solution of the differential equation

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} + [\nu^2(M^2-1) - \frac{n^2}{r^2}]\, f(r) = 0 \qquad (1.5)$$

and satisfies boundary conditions corresponding to the conditions (1.3). Knowing the velocity potential ϕ one obtains the resulting pressure perturbation $\Delta p(x,r,\theta)$ by (see e.g. ref. 4, page 432)

$$\Delta p(x,r,\theta) = -\rho_0\, a_0\, M \frac{\partial\phi}{\partial x} . \qquad (1.6)$$

In this manner one arrives at the following results for the pressure perturbation acting on the cylinder well, $\Delta p = \Delta p(x,R,\theta)$, which depends upon the Mach number range under consideration (for details the reader is referred to ref. 9):[1]

$$\Delta p = \frac{\rho_0 (a_0 M)^2}{\sqrt{1-M^2}}\ \frac{\nu^2}{|\nu|}\ \frac{K_n(|\nu|R\sqrt{1-M^2})}{K_n'(|\nu|R\sqrt{1-M^2})}\, w(x,\theta) \qquad (1.7a)$$

for $-1 < M < 1$,

[1] Here and in the following we always choose the positive square-root.

$$\Delta p = \frac{\rho_0 (a_0 M)^2}{\sqrt{M^2-1}} \; \frac{\nu^2}{|\nu|} \; \frac{H_n^{(1)}(|\nu| R \sqrt{M^2-1})}{H_n^{(1)'}(|\nu| R \sqrt{M^2-1})} \; w(x,\theta) \tag{1.7b}$$

for $|M| > 1$ and $M\nu > 0$,

$$\Delta p = \frac{\rho_0 (a_0 M)^2}{\sqrt{M^2-1}} \; \frac{\nu^2}{|\nu|} \; \frac{H_n^{(2)}(|\nu| R \sqrt{M^2-1})}{H_n^{(2)'}(|\nu| R \sqrt{M^2-1})} \; w(x,\theta) \tag{1.7c}$$

for $|M| > 1$ and $M\nu < 0$.

Here $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$ are the Hankel functions (Bessel functions of the third kind) and $K_n(z)$ the modified Hankel functions (modified Bessel functions of the third kind). Because $K_n(z)$ is real-valued when n is real and z is positive (see e.g. ref. 3), which is obviously the case here, the perturbation pressure Δp for $|M| < 1$ is "in phase" with the shell deformation $w(x,\theta)$, see eq. (1.7a). This is no longer true for $|M| > 1$. For the fractions of the Hankel-functions and their derivatives involved in eqs. (1.7b) and (1.7c) the following relation holds

$$\frac{H_n^{(1)}(|\nu| R \sqrt{M^2-1})}{H_n^{(1)'}(|\nu| R \sqrt{M^2-1})} = \overline{\left(\frac{H_n^{(2)}(|\nu| R \sqrt{M^2-1})}{H_n^{(2)'}(|\nu| R \sqrt{M^2-1})}\right)} \tag{1.8}$$

for $|M| > 1$. [1]

[1] Here and in the following the barred symbol denotes the conjugate complex number, i.e. $\overline{a+ib} = a-ib$; (a, b real).

2. Aerodynamic Pressure Acting on a Shell which is Deformed by a Traveling Sinusoidal Wave or by a Standing Sinusoidal Wave

Now we consider shell deformation of the form

$$w(x,\theta,t) = w_0 \cos(n\theta)\, e^{-i\nu x} e^{i\omega t} = \\ = w_0 \cos(n\theta)\, e^{-i\nu(x - \frac{\omega}{\nu}t)}, \quad (\omega \geq 0) \tag{2.1}$$

where ω is assumed to be a real, non-negative number. The deformation w represents a sinusoidal wave which travels with the velocity $\frac{\omega}{\nu}$ in the positive x-direction. A moving observer who travels with the same velocity $\frac{\omega}{\nu}$ in the positive x-direction can apply the formulas (1.7) after M has been replaced by

$$M_1 = M - \frac{\omega}{\nu a_0} \; . \tag{2.2}$$

Referring these results again to the (resting) x, r, θ-coordinate system one obtains for the pressure perturbation acting on the vibrating cylinder wall

$$\Delta p = \frac{\rho_0 (a_0 M_1)^2}{\sqrt{1 - M_1^2}} \, \frac{\nu^2}{|\nu|} \, \frac{K_n(|\nu| R\sqrt{1-M_1^2})}{K_n'(|\nu| R\sqrt{1-M_1^2})} \, w_0 \cos(n\theta)\, e^{-i\nu(x - \frac{\omega}{\nu}t)} \\ \text{for} \quad |M_1| < 1, \tag{2.3a}$$

$$\Delta p = \frac{\rho_0 (a_0 M_1)^2}{\sqrt{M_1^2 - 1}} \, \frac{\nu^2}{|\nu|} \, \frac{H_n^{(1)}(|\nu| R\sqrt{M_1^2-1})}{H_n^{(1)'}(|\nu| R\sqrt{M_1^2-1})} \, w_0 \cos(n\theta) e^{-i\nu(x - \frac{\omega}{\nu}t)} \tag{2.3b}$$

$$\text{for} \quad |M_1| > 1 \;\text{ and }\; M_1\nu > 0 \, ,$$

$$\Delta p = \frac{\rho_0 (a_0 M_1)^2}{\sqrt{M_1^2 - 1}} \frac{\nu^2}{|\nu|} \frac{H_n^{(2)}(|\nu| R\sqrt{M_1^2-1})}{H_n^{(2)'}(|\nu| R\sqrt{M_1^2-1})} w_0 \cos(n\theta) e^{-i\nu(x - \frac{\omega}{\nu} t)} \tag{2.3c}$$

for $|M_1| > 1$ and $M_1\nu < 0$.

Shell deformations of the form

$$w(x, \theta, t) = w_0 \cos(n\theta)\, s(x)\, e^{i\omega t} \tag{2.4a}$$

with

$$s(x) = \begin{cases} \sin(\nu x) \\ \text{or} \\ \cos(\nu x)\ , \end{cases} \tag{2.4b}$$

which represent a standing sinusoidal wave, are obtained by suitable superpositions of two waves of the form (2.1), traveling in opposite directions. Hence the corresponding aerodynamic pressure is a superposition of the corresponding pressure terms (2.3). Introducing the notation

$$M_2 = M + \frac{\omega}{\nu a_0} \tag{2.5}$$

we find especially:

$$\Delta p = i w_0 \cos(n\theta) e^{i\omega t} \rho_0 a_0^2 \frac{\nu}{2} \left\{ \frac{M_1^2}{\sqrt{M_1^2-1}} \frac{H_n^{(1)}(|\nu| R\sqrt{M_1^2-1})}{H_n^{(1)'}(|\nu| R\sqrt{M_1^2-1})} e^{-i\nu x} - \frac{M_2^2}{\sqrt{M_2^2-1}} \frac{H_n^{(2)}(|\nu| R\sqrt{M_2^2-1})}{H_n^{(2)'}(|\nu| R\sqrt{M_2^2-1})} e^{i\nu x} \right\} \tag{2.6a}$$

for

$$M_1 > 1;\ M_2 > 1;\ \nu > 0;\ w = w_0 \cos(n\theta)\ \sin(\nu x) e^{i\omega t}; \tag{2.6b}$$

$$\Delta p = i w_0 \cos(n\theta) e^{i\omega t} \rho_0 a_0^2 \frac{|\nu|}{2} \left\{ \frac{M_1^2}{\sqrt{M_1^2-1}} \frac{H_n^{(2)}(|\nu| R \sqrt{M_1^2-1})}{H_n^{(2)'}(|\nu| R \sqrt{M_1^2-1})} e^{-i\nu x} - \right.$$

$$\left. - \frac{M_2^2}{\sqrt{M_2^2-1}} \frac{H_n^{(1)}(|\nu| R \sqrt{M_2^2-1})}{H_n^{(2)'}(|\nu| R \sqrt{M_2^2-1})} e^{i\nu x} \right\} \tag{2.7a}$$

for

$$M_1 > 1;\ M_2 > 1;\ \nu < 0;\ w = w_0 \cos(n\theta) \sin(\nu x) e^{i\omega t}; \tag{2.7b}$$

$$\Delta p = w_0 \cos(n\theta) e^{i\omega t} \rho_0 a_0^2 \frac{\nu}{2} \left\{ \frac{M_1^2}{\sqrt{M_1^2-1}} \frac{H_n^{(1)}(|\nu| R \sqrt{M_1^2-1})}{H_n^{(1)'}(|\nu| R \sqrt{M_1^2-1})} e^{-i\nu x} + \right.$$

$$\left. + \frac{M_2^2}{\sqrt{M_2^2-1}} \frac{H_n^{(2)}(|\nu| R \sqrt{M_2^2-1})}{H_n^{(2)'}(|\nu| R \sqrt{M_2^2-1})} e^{i\nu x} \right\} \tag{2.8a}$$

for

$$M_1 > 1;\ M_2 > 1;\ \nu > 0;\ w = w_0 \cos(n\theta) \cos(\nu x) e^{i\omega t}; \tag{2.8b}$$

$$\Delta p = i w_0 \cos(n\theta) e^{i\omega t} \rho_0 a_0^2 \frac{\nu}{2} \left\{ \frac{M_1^2}{\sqrt{M_1^2-1}} \frac{H_n^{(2)}(|\nu| R \sqrt{M_1^2-1})}{H_n^{(2)'}(|\nu| R \sqrt{M_1^2-1})} e^{-i\nu x} - \right.$$

$$\left. - \frac{M_2^2}{\sqrt{M_2^2-1}} \frac{H_n^{(2)}(|\nu| R \sqrt{M_2^2-1})}{H_n^{(2)'}(|\nu| R \sqrt{M_2^2-1})} e^{i\nu x} \right\} \tag{2.9a}$$

for

$$M_1 < -1;\ M_2 > 1;\ \nu > 0;\ w = w_0 \cos(n\theta)\sin(\nu x)e^{i\omega t}; \tag{2.9b}$$

$$\Delta p = w_0 \cos(n\theta)e^{i\omega t}\rho_0 a_0^2 \frac{|\nu|}{2}\left\{\frac{M_1^2}{\sqrt{M_1^2-1}}\,\frac{H_n^{(2)}(|\nu|R\sqrt{M_1^2-1})}{H_n^{(2)'}(|\nu|R\sqrt{M_1^2-1})}\,e^{-i\nu x} + \right.$$

$$\left. + \frac{M_2^2}{\sqrt{M_2^2-1}}\,\frac{H_n^{(2)}(|\nu|R\sqrt{M_2^2-1})}{H_n^{(2)'}(|\nu|R\sqrt{M_2^2-1})}\,e^{i\nu x}\right\} \tag{2.10a}$$

for

$$M_1 > 1;\ M_2 < -1;\ \nu < 0;\ w = w_0 \cos(n\theta)\cos(\nu x)e^{i\omega t}; \tag{2.10b}$$

$$\Delta p = i w_0 \cos(n\theta)e^{i\omega t}\rho_0 a_0^2 \frac{|\nu|}{2}\left\{\frac{M_1^2}{\sqrt{1-M_1^2}}\,\frac{K_n(|\nu|R\sqrt{1-M_1^2})}{K_n'(|\nu|R\sqrt{1-M_1})}\,e^{-i\nu x} - \right.$$

$$\left. - \frac{M_2^2}{\sqrt{M_2^2-1}}\,\frac{K_n(|\nu|R\sqrt{1-M_2^2})}{K_n'(|\nu|R\sqrt{1-M_2^2})}\,e^{i\nu x}\right\} \tag{2.11a}$$

for

$$|M_1| < 1;\quad |M_2| < 1;\ \nu \gtrless 0;\ w = w_0 \cos(n\theta)\sin(\nu x)e^{i\omega t}; \tag{2.11b}$$

$$\Delta p = i w_0 \cos(n\theta)e^{i\omega t}\rho_0 a_0^2 \frac{\nu}{2}\left\{\frac{M_1^2}{\sqrt{1-M_1^2}}\,\frac{K_n(|\nu|R\sqrt{1-M_1^2})}{K_n'(|\nu|R\sqrt{1-M_1^2})}\,e^{-i\nu x} - \right.$$

$$\left. - \frac{M_2^2}{\sqrt{M_2^2-1}}\,\frac{H_n^{(2)}(|\nu|R\sqrt{M_2^2-1})}{H_n^{(2)'}(|\nu|R\sqrt{M_2^2-1})}\,e^{i\nu x}\right\} \tag{2.12a}$$

for

$$|M_1| < 1;\ M_2 > 1;\ \nu > 0;\ w = w_0 \cos(n\theta) \sin(\nu x)\, e^{i\omega t}\ . \tag{2.12b}$$

Letting the Mach number M tend to zero, one obtains the aerodynamic pressure acting on the vibrating cylinder, which performs the vibration (2.4a) in still air. This can be realized by inspection of the formulas (2.9) and (2.11) and by a comparison with the corresponding results in ref. 8. (Notice: The positive w-direction in ref. 8 is opposite to the one chosen in the present paper.)

3. Asymptotic Expansions of the Involved Cylinder Function Terms

In this section we apply the following well-known asymptotic expansions for cylinder functions for $\xi \to \infty$ (see e.g. ref. 3, pages 85 and 86).[1]

$$H_n^{(1)}(\xi) = \sqrt{\frac{2}{\pi\xi}}\, e^{i(\xi - \frac{n\pi}{2} - \frac{\pi}{4})} \left\{ \sum_{q=0}^{Q-1} \frac{(n,q)}{(-2i\xi)^q} + O(\xi^{-Q}) \right\}, \qquad (3.1)$$

$$K_n(\xi) = \sqrt{\frac{\pi}{2\xi}}\, e^{-\xi} \left\{ \sum_{q=0}^{Q-1} \frac{(n,q)}{(2\xi)^q} + O(\xi^{-Q}) \right\}, \qquad (3.2)$$

with

$$\begin{aligned} &(n,0) = 1; \\ &(n,q) = (2^{2q} q!)^{-1} (4n^2-1)(4n^2-3^2) \ldots (4n^2-[2q-1]^2); \end{aligned} \qquad (3.3)$$

to the exact expressions for the aerodynamic pressure stated in section 2. Using eq. (3.1) one arrives in a straightforward manner at

$$\frac{H_n^{(1)}(\xi)}{H_n^{(1)'}(\xi)} = -i - \tfrac{1}{2}\xi^{-1} - i\left(\frac{n^2}{2} - \frac{3}{8}\right)\xi^{-2} - \left(n^2 - \frac{3}{8}\right)\xi^{-3} + \qquad (3.4)$$
$$+ O(\xi^{-4}) \text{ for } \xi \to +\infty.$$

Because ξ is assumed to be real valued, one obtains (see eq. (1.8))

$$\frac{H_n^{(2)}(\xi)}{H_n^{(2)'}(\xi)} = \overline{\left(\frac{H_n^{(1)}(\xi)}{H_n^{(1)'}(\xi)}\right)} = i - \tfrac{1}{2}\xi^{-1} + i\left(\frac{n^2}{2} - \frac{3}{8}\right)\xi^{-2} - \left(n^2 - \frac{3}{8}\right)\xi^{-3} + \qquad (3.5)$$
$$+ O(\xi^{-4}) \text{ for } \xi \to \infty$$

[1] Here and in the following it is assumed that ξ is a real variable. O is the Landau-symbol; i.e. $F(\xi) = O(\xi^{-Q})$ for $\xi \to \infty$ means that $F(\xi)\xi^{Q}$ is bounded as ξ tends to infinity.

Further, applying eq. (3.2) one finds

$$\frac{K_n(\xi)}{K_n'(\xi)} = -1 + \tfrac{1}{2}\,\xi^{-1} + \left(\frac{n^2}{2} - \frac{3}{8}\right)\xi^{-2} - \left(n^2 - \frac{3}{8}\right)\xi^{-3} + O(\xi^{-4}) \qquad (3.6)$$

$$\text{for} \quad \xi \to \infty .$$

From the well-known relations (see e.g. ref. 3)

$$H_n^{(1)}(\xi) = (-1)^n H_{-n}^{(1)}(\xi); \quad H_n^{(2)}(\xi) = (-1)^n H_{-n}^{(2)}(\xi);$$

$$K_n(\xi) = K_{-n}(\xi) \qquad (3.7)$$

one learns the following: If one includes in the asymptotic expansions (3.4), (3.5) and (3.6) higher powers of $\frac{1}{\xi}$, then the corresponding coefficients depend on powers of n with even exponents.

Table I on page 28 gives some information about the accuracy of the expansion (3.4). The parameter values R and ν are chosen with respect to an electroplated copper cylinder which was used repeatedly during the flutter experiments, see ref. 11. The length of this cylinder is $L = 16$ inches, its radius is $R = 8$ inches.

Using the expansions (3.4) and (3.5) one realizes immediately that for $R \to \infty$ the aerodynamic pressures (1.7b) and (1.7c) (i.e. the aerodynamic pressure acting on a stationary and sinusoidally deformed cylinder for $|M| > 1$) tends to Ackeret's formula

$$\Delta p^* = \frac{\rho_0 (a_0 M)^2}{\sqrt{M^2 - 1}} \; \frac{d(w_0\, e^{-i\nu x})}{dx} . \qquad (3.8)$$

4. Asymptotic Expansion of the Aerodynamic Pressure for $|M_1| > 1$ and $|M_2| > 1$.

Let the cylinder deformation w be given by eq. (2.4a) and let M_1 and M_2 be restricted by $|M_1| > 1$ and $|M_2| > 1$. We apply the results of section 3 to the exact expressions for the aerodynamic pressure, which are partially listed in eqs. (2.6) to (2.10). We obtain

$$
\begin{aligned}
\Delta p = a_0 \rho_0 \, & \{a_0 M \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t}\} + \\
+ \tfrac{1}{2} a_0^2 \rho_0 \Big\{ & i\nu \Big(- \frac{M_1}{[1-\frac{1}{M_1^2}]^{\frac{1}{2}}} + \frac{M_2}{[1-\frac{1}{M_2^2}]^{\frac{1}{2}}} - \frac{2\omega}{\nu a_0} \Big) w + \Big(\frac{M_1}{[1-\frac{1}{M_1^2}]^{\frac{1}{2}}} + \frac{M_2}{[1-\frac{1}{M_2^2}]^{\frac{1}{2}}} - 2M \Big) \frac{\partial w}{\partial x} + \\
& + \frac{1}{2R} \Big(- \frac{1}{1-\frac{1}{M_1^2}} - \frac{1}{1-\frac{1}{M_2^2}} \Big) w + \frac{i}{2\nu R} \Big(- \frac{1}{1-\frac{1}{M_1^2}} + \frac{1}{1-\frac{1}{M_2^2}} \Big) \frac{\partial w}{\partial x} + \\
& + i \frac{n^2 - \frac{3}{4}}{2\nu R^2} \Big(- \frac{1}{M_1[1-\frac{1}{M_1^2}]^{3/2}} + \frac{1}{M_2[1-\frac{1}{M_2^2}]^{3/2}} \Big) w + \\
& + \frac{n^2 - \frac{3}{4}}{2\nu^2 R^2} \Big(\frac{1}{M_1[1-\frac{1}{M_1^2}]^{3/2}} + \frac{1}{M_2[1-\frac{1}{M_2^2}]^{3/2}} \Big) \frac{\partial w}{\partial x} + \\
& + \frac{n^2 - \frac{3}{8}}{\nu^2 R^3} \Big(- \frac{1}{M_1^2 [1-\frac{1}{M_1^2}]^2} - \frac{1}{M_2^2 [1-\frac{1}{M_2^2}]^2} \Big) w + \\
& + i \frac{n^2 - \frac{3}{8}}{\nu^3 R^3} \Big(- \frac{1}{M_1^2 [1-\frac{1}{M_1^2}]^2} + \frac{1}{M_2^2 [1-\frac{1}{M_2^2}]^2} \Big) \frac{\partial w}{\partial x} + \\
& + w \Big[O\Big(\frac{1}{\nu^3 R^4} \frac{1}{M_1^3 [1-\frac{1}{M_1^2}]^{5/2}} \Big) + O\Big(\frac{1}{\nu^3 R^4} \frac{1}{M_2^3 [1-\frac{1}{M_2^2}]^{5/2}} \Big) \Big] + \\
& + \frac{\partial w}{\partial x} \Big[O\Big(\frac{1}{\nu^4 R^4} \frac{1}{M_1^3 [1-\frac{1}{M_1^2}]^{5/2}} \Big) + O\Big(\frac{1}{\nu^4 R^4} \frac{1}{M_2^3 [1-\frac{1}{M_2^2}]^{5/2}} \Big) \Big] \Big\}
\end{aligned}
\tag{4.1a}
$$

with

$$M_1 = M - \frac{\omega}{\nu a_0}\ ; \quad M_2 = M + \frac{\omega}{\nu a_0} \tag{4.1b}$$

and

$$w = w_0 \cos(n\theta) s(x) e^{i\omega t};\quad s(x) = \begin{cases} \sin(\nu x) \\ \text{or} & \nu \gtrless 0 \\ \cos(\nu x) \end{cases} \tag{4.1c}$$

for

$$|M_1| > 1; \qquad |M_2| > 1. \tag{4.1d}$$

(Notice: For $M_1 < -1$ it holds $M_1^2/[M_1^2-1]^{\frac{1}{2}} = -M_1/[1 - \frac{1}{M_1^2}]^{\frac{1}{2}}$.) The first term on the right hand side of eq. (4.1a) is the approximation by means of the linear piston theory

$$\Delta p^{**} = a_0 \rho_0 \left\{ a_0 M \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} \right\}, \tag{4.2}$$

which has been separated. The other terms on the right side of eq. (4.1a) are the "correction terms" and a "remainder", which demonstrate how far the linear piston theory approximation (4.2) is in error. The first two correction terms are

$$\tfrac{1}{2} a_0^2 \rho_0 i\nu \left(- \frac{M_1}{[1 - \frac{1}{M_1^2}]^{\frac{1}{2}}} + \frac{M_2}{[1 - \frac{1}{M_2^2}]^{\frac{1}{2}}} - \frac{2\omega}{\nu a_0} \right) w = F_0\ w \tag{4.3a}$$

and

$$\tfrac{1}{2} a_0^2 \rho_0 \left(\frac{M_1}{[1 - \frac{1}{M_1^2}]^{\frac{1}{2}}} + \frac{M_2}{[1 - \frac{1}{M_2^2}]^{\frac{1}{2}}} - 2M \right) \frac{\partial w}{\partial x} = F_0^* \frac{\partial w}{\partial x}. \tag{4.3b}$$

The remaining correction terms are of the form

$$\tfrac{1}{2}a_0^2\rho_0\,\frac{A_m}{\nu^{m-1}R^m}\left(\frac{1}{M_1^{m-1}[1-\frac{1}{M_1^2}]^{(m+1)/2}}\pm\frac{1}{M_2^{m-1}[1-\frac{1}{M_2^2}]^{(m+1)/2}}\right)w = \frac{F_m}{R^m}\,w \tag{4.3c}$$

and

$$\tfrac{1}{2}a_0^2\rho_0\,\frac{A_m^*}{\nu^{m}R^m}\left(\frac{1}{M_1^{m-1}[1-\frac{1}{M_1^2}]^{(m+1)/2}}\pm\frac{1}{M_2^{m-1}[1-\frac{1}{M_2^2}]^{(m+1)/2}}\right)\frac{\partial w}{\partial x} = \frac{F_m^*}{R^m}\frac{\partial w}{\partial x}, \tag{4.3d}$$

$m = 1,\ 2,\ 3.$

The coefficients A_m and A_m^*, $m = 2, 3$, are polynomials in n^2, see section 3. By means of eqs. (4.3) new coefficients F_m and F_m^*, $m = 0, 1, 2, 3$, are introduced. Naturally, always only one of the two signs in (4.3c) and (4.3d) is relevant. Expanding the coefficients F_m and F_m^* in power series of $\frac{1}{M_1}$ and $\frac{1}{M_2}$ one obtains, after some regrouping [1)]

$$F_0 = -\frac{i}{2}\,a_0^2\rho_0\nu\left\{\frac{\frac{\omega}{\nu a_0}}{M_1 M_2} + \left|\binom{-\frac{1}{2}}{2}\right|\frac{M_2^3 - M_1^3}{M_1^3M_2^3} + \left|\binom{-\frac{1}{2}}{3}\right|\frac{M_2^5-M_1^5}{M_1^5M_2^5} + \ldots\right\}, \tag{4.4a}$$

---

1) $\left|\binom{-\frac{1}{2}}{2}\right|$ denotes the absolute value of the binomial-coefficient $\binom{-\frac{1}{2}}{2}$. The same holds for the following.

$$F_0^* = \tfrac{1}{2}a_0^2\rho_0 \left\{\frac{M}{M_1M_2} + \left|\binom{-\frac{1}{2}}{2}\right| \frac{M_2^3+M_1^3}{M_1^3M_2^3} + \left|\binom{-\frac{1}{2}}{3}\right| \frac{M_2^5-M_1^5}{M_1^5M_2^5} + \ldots\right\}, \quad (4.4b)$$

$$F_1 = -\frac{1}{4} a_0^2\rho_0 \left\{2 + \frac{M_2^2+M_1^2}{M_1^2M_2^2} + \frac{M_2^4+M_1^4}{M_1^4\,M_2^4} + \ldots\ldots\right\} , \quad (4.4c)$$

$$F_1^* = -\frac{i}{4\nu} a_0^2\rho_0 \left\{ \frac{4M\frac{\omega}{\nu a_0}}{M_1^2\,M_2^2} + \frac{M_2^4-M_1^4}{M_1^4\,M_2^4} + \frac{M_2^6-M_1^6}{M_1^6\,M_2^6} + \ldots\right\} , \quad (4.4d)$$

$$F_m = \tfrac{1}{2}a_0^2\rho_0 \frac{A_m}{\nu^{m-1}} \left\{ \frac{M_2^{m-1} \pm M_1^{m-1}}{M_1^{m-1}M_2^{m-1}} + \left|\binom{-\frac{m+1}{2}}{1}\right| \frac{M_2^{m+1} \pm M_1^{m+1}}{M_1^{m+1}M_2^{m+1}} + \right.$$
$$\left. + \left|\binom{-\frac{m+1}{2}}{2}\right| \frac{M_2^{m+3} \pm M_1^{m+3}}{M_1^{m+3}M_2^{m+3}} + \ldots\right\} , \quad (4.4e)$$

$$F_m^* = \tfrac{1}{2}a_0^2\rho_0 \frac{A_m}{\nu^{m}} \left\{ \frac{M_2^{m-1} \pm M_1^{m-1}}{M_1^{m-1}M_2^{m-1}} + \left|\binom{-\frac{m+1}{2}}{1}\right| \frac{M_2^{m+1} \pm M_1^{m+1}}{M_1^{m+1}M_2^{m+1}} + \right.$$
$$\left. + \left|\binom{-\frac{m+1}{2}}{2}\right| \frac{M_2^{m+3} \pm M_1^{m+3}}{M_1^{m+3}M_2^{m+3}} + \ldots\right\} , \quad (4.4f)$$

Furthermore it holds

$$M_2^\mu + M_1^\mu = 2\left\{M^\mu + \binom{\mu}{2} M^{\mu-2}\left(\frac{\omega}{\nu a_0}\right)^2 + \ldots\right\} , \quad (4.5a)$$

$$M_2^\mu - M_1^\mu = 2\,\frac{\omega}{\nu a_0} \left\{\binom{\mu}{1} M^{\mu-1} + \binom{\mu}{3} M^{\mu-3}\left(\frac{\omega}{\nu a_0}\right)^2 + \ldots\right\} \quad (4.5b)$$

$$M_1^\mu M_2^\mu = \left\{M^2 - \left(\frac{\omega}{\nu a_0}\right)^2\right\}^\mu , \quad (4.5c)$$

$$\mu = 1, 2, \ldots ,$$

where, of course, the dots in (4.5a) and (4.5b) stand for finite sums.

Using the coefficients F_m and F_m^*, introduced by eqs. (4.3), one can rewrite eq. (4.1a) in the form

$$\Delta p = a_0\rho_0 \left\{ a_0 \; M \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} \right\} + \tag{4.6}$$

$$+ F_0 w + F_0^* \frac{\partial w}{\partial x} + \frac{1}{R}\left(F_1 w + F_1^* \frac{\partial w}{\partial x}\right) +$$

$$+ \frac{1}{R^2}\left(F_2 w + F_2^* \frac{\partial w}{\partial x}\right) + \frac{1}{R^3}\left(F_3 w + F_3^* \frac{\partial w}{\partial x}\right) +$$

$$+ w\left[O\left(\frac{1}{\nu^3 R^4} \frac{1}{M_1^3[1 - \frac{1}{M_1^2}]^{5/2}}\right) + O\left(\frac{1}{\nu^3 R^4} \frac{1}{M_2^3[1 - \frac{1}{M_2^2}]^{5/2}}\right)\right] +$$

$$\frac{\partial w}{\partial x}\left[O\left(\frac{1}{\nu^4 R^4} \frac{1}{M_1^3[1 - \frac{1}{M_1^2}]^{5/2}}\right) + O\left(\frac{1}{\nu^4 R^4} \frac{1}{M_2^3[1 - \frac{1}{M_2^2}]^{5/2}}\right)\right].$$

Now we are ready to discuss the behavior of the correction terms and the remainder as one of the parameters $|M|$, R, ω, and $|\nu|$ tend to $+\infty$, provided that the inequalities (4.1d) are observed. First of all an inspection of the remainder in eq. (4.1a) or eq. (4.6) discloses immediately that the remainder tends always to zero as soon as at least one of the above mentioned four parameters tends to $+\infty$.

The following statements are based on eqs. (4.1) to (4.6) and the just quoted statement concerning the behavior of the remainder. For $|M| \to \infty$ we obtain

$$F_m \to 0 \text{ for } m = 0,2,3; \; F_m^* \to 0 \text{ for } m = 0,1,2,3;$$
$$F_1 \to -\tfrac{1}{2} a_0^2 \rho_0 \text{ for } |M| \to \infty, \tag{4.7a}$$

hence

$$\Delta p - \{\Delta p^{**} - \frac{a_0^2 \rho_0}{2R} w\} \to 0 \quad \text{for} \quad |M| \to \infty. \tag{4.7b}$$

For $R \to \infty$ it holds

$$\frac{F_m}{R_m} \to 0; \quad \frac{F_m^*}{R^m} \to 0 \quad \text{for} \quad R \to \infty,\ m = 1,2,3, \tag{4.8a}$$

hence

$$\Delta p - \Bigg[\Delta p^{**} - \tfrac{1}{2} a_0 \rho_0 \Bigg\{ \frac{1}{M_1 M_2} + \left|\binom{-\frac{1}{2}}{2}\right| \frac{\frac{\nu a_0}{\omega}(M_2^3 - M_1^3)}{M_1^3 M_2^3} +$$

$$+ \left|\binom{-\frac{1}{2}}{3}\right| \frac{\frac{\nu a_0}{\omega}(M_2^5 - M_1^5)}{M_1^5 M_2^5} + \dots \Bigg\} \frac{\partial w}{\partial t} +$$

$$+ \tfrac{1}{2} a_0^2 \rho_0 M \Bigg\{ \frac{1}{M_1 M_2} + \left|\binom{\frac{1}{2}}{2}\right| \frac{M_2^3 + M_1^3}{M M_1^3 M_2^3} + \left|\binom{\frac{1}{2}}{3}\right| \frac{M_2^5 + M_1^5}{M M_1^5 M_2^5} + \dots \Bigg\} \frac{\partial w}{\partial x} \Bigg] \to 0 \tag{4.8b}$$

for $R \to \infty$.

From eqs. (4.7) and (4.8) we conclude

$$\Delta p - \Delta p^{**} \to 0 \quad \text{for} \quad |M| \to \infty,\ R \to \infty. \tag{4.9}$$

Further we obtain

$$F_m \to 0,\ m = 0,2,3; \quad F_m^* \to 0,\ m = 0,1,2,3; \quad F_1 \to -\tfrac{1}{2} a_0^2 \rho_0$$

$$\text{for} \quad \omega \to +\infty, \tag{4.10a}$$

hence

$$\Delta p - \{\Delta p^{**} - \frac{a_0^2 \rho_0}{2R} w\} \to 0 \text{ for } \omega \to +\infty. \tag{4.10b}$$

Eqs. (4.8) and (4.10) lead to

$$\Delta p - \Delta p^{**} \to 0 \text{ for } R \to \infty,\ \omega \to \infty. \tag{4.11}$$

In order to investigate the limiting process $|\nu| \to \infty$ we restrict M by

$$|M| > 1. \tag{4.12}$$

Thanks to this inequality we are assured that for sufficiently large values of $|\nu|$ the inequalities (4.1d) are satisfied. We obtain

$$F_0 \to i\omega a_0 \rho_0 \left\{ \frac{|M|}{(M^2-1)^{\frac{1}{2}}} \frac{M^2-2}{M^2-1} - 1 \right\};\ F_0^* \to a_0^2 \rho_0 \left\{ \frac{M|M|}{(M^2-1)^{\frac{1}{2}}} - M \right\},$$

$$F_1 \to -\frac{a_0^2 \rho_0}{2} \frac{M^2}{M^2-1};\ F_m \to 0,\ m = 2,3;\ F_m^* \to 0,\ m = 1,2,3 \tag{4.13a}$$

$$\text{for } |\nu| \to \infty.$$

These relations lead to

$$\Delta p - \left\{ \Delta p^{**} + a_0 \rho_0 \left(\frac{|M|}{(M^2-1)^{\frac{1}{2}}} \frac{M^2-2}{M^2-1} - 1 \right) \frac{\partial w}{\partial t} + a_0^2 \rho_0 \left(\frac{M|M|}{(M^2-1)^{\frac{1}{2}}} - M \right) \frac{\partial w}{\partial x} - \frac{a_0^2 \rho_0}{2R} \frac{M^2}{M^2-1} w \right\} \to 0 \text{ for } |\nu| \to \infty. \tag{4.13b}$$

From the relation (4.13b) we obtain immediately

$$\Delta p - \{\Delta p^{**} - \frac{a_0^2 \rho_0}{2R} w\} \to 0 \text{ for } |M| \to \infty,\ |\nu| \to \infty, \tag{4.13c}$$

$$\Delta p - \Delta p^{**} \rightarrow 0 \text{ for } |M| \rightarrow \infty, \ |\nu| \rightarrow \infty, \ R \rightarrow \infty. \tag{4.13d}$$

In case the inequality (4.12) does not hold, the restrictions (4.1d) would be violated sooner or later as $|\nu|$ tends to infinity, no matter how large the value of ω is. Then the expansion (4.1a) is no longer valid. This case will be dealt with shortly in section 5.

It should be emphasized that the coefficient $-\frac{a_0^2 \rho_0}{2R}$ of the correction term $-\frac{a_0^2 \rho_0}{2R} w$, which appears especially in eqs. (4.7b), (4.10b), and (4.13c) is independent of the parameters M, ω, ν, and n.

It is almost superfluous to quote the following remark: The asymptotic expansions in section 3 were carried out up to the term of the order ξ^{-3}. In accordance with this the expansions (4.1a) and (4.6) are extended up to the term of the order R^{-3}. In case the expansions would be carried out to higher order terms, one would obtain expansions for the aerodynamic pressure analogous to (4.1a) and (4.6). Formulas (4.3c), (4.3d), (4.4e), (4.4f), (4.7) to (4.13) would be valid too, after the range of the index m is extended appropriately.

From eqs. (4.1a), (3.4), and (3.5) one learns that the coefficients of $(1/R)^2$, $(1/R)^3$ in the asymptotic expansion for Δp depend upon n^2. (The same is true for the coefficients of higher powers of $1/R$, if the asymptotic expansion would be carried out to higher terms, see section 3.) For the investigation of the flutter of cylindrical shells it is often necessary to consider values of n up to the order of 20, see refs. 1 and 6. For these large values of n the influence of the terms of the order $(1/R)^2$ and $(1/R)^3$ in (4.1a) and (4.6) can become quite significant.

5. Examination of the Remaining Cases.

In section 4 only the case $|M_1| > 1$, $|M_2| > 1$ has been considered, where the linear piston theory expression proved to be a first order approximation for the aerodynamic pressure under investigation. In the remaining case, i.e.

$$\text{a.)}\ |M_1| < 1;\ |M_2| < 1; \qquad \text{b.)}\ |M_1| < 1;\ |M_2| > 1;$$
$$\text{c.)}\ |M_1| > 1;\ |M_2| < 1 \tag{5.1}$$

the formulas (2.11), (2.12) and similar ones have to be applied, and besides the expansions (3.4) and (3.5) also the asymptotic expansion (3.6) has to be used. In none of these remaining cases (5.1) the linear piston theory expression (4.2) can be considered as a first order approximation of the aerodynamic pressure under investigation. Let w, the cylinder deformation, again be given by eq. (2.4). Then one obtains for instance:

$$\Delta p - \Delta p^{**} =$$
$$= a_0^2 \rho_0 \left\{ i \left(iM + 2M \frac{\omega}{\nu a_0} + \frac{1}{4} [M_2^4 - M_1^4] + \dots \right) \frac{\partial w}{\partial x} - \right. \tag{5.2a}$$
$$\left. -\nu \left(\frac{i\omega}{\nu a_0} + M^2 + [\frac{\omega}{\nu a_0}]^2 + \frac{1}{4} [M_2^4 + M_1^4] + \dots \right) w + O(R^{-1}) \right\}$$

for

$$|M_1| < 1; \quad |M_2| < 1; \quad \nu > 0 \tag{5.2b}$$

and

$$\Delta p - \Delta p^{**} =$$

$$= \tfrac{1}{2}\, a_0^2 \rho_0 \Big\{ -\nu \Big(M_1^2 + \tfrac{1}{2} M_1^4 + \dots\Big) w + i\nu \Big(M - \frac{\omega}{\nu a_0} + \tfrac{1}{2}\frac{1}{M_2} + \frac{3}{8}\frac{1}{M_2^3} + \dots\Big) w -$$
$$- i \Big(M_1^2 + \tfrac{1}{2} M_1^4 + \dots\Big) \frac{\partial w}{\partial x} + \Big(-M + \frac{\omega}{\nu a_0} + \tfrac{1}{2}\frac{1}{M_2} + \frac{3}{8}\frac{1}{M_2^3} + \dots\Big) \frac{\partial w}{\partial x} +$$
$$+ O(R^{-1}) \Big\} \qquad (5.3a)$$

for

$$|M_1| < 1; \quad M_2 > 1; \quad \nu > 0 \, . \qquad (5.3b)$$

6. Conclusions.

In the preceding sections the aerodynamic pressure acting on a circular cylindrical shell of infinite length in an airflow parallel to the cylinder axis is investigated. The shell is deformed by a harmonically oscillating standing wave of a sinusoidal pattern. Based upon the exact solutions, furnished by Leonhard and Hedgepeth, asymptotic expansions are developed for the aerodynamic pressure. These expansions allow an investigation of the accuracy of the linear piston theory approximation (4.2) when applied to cylindrical shells. Furthermore improved approximations can be obtained from these asymptotic expansions.

In case the absolute values of M_1 and M_2 (see eqs. (2.2) and (2.5)) are larger than one, $|M_1| > 1$; $|M_2| > 1$, then the linear piston theory expression Δp^{**} can be considered as a first order approximation for the aerodynamic pressure under investigation. Nevertheless the replacement of Δp^{**} by the approximation

$$\Delta p^{***} = \Delta p^{**} - \frac{a_0^2 \rho_0}{2R} w = a_0 \rho_0 \left\{ a_0 M \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} - \frac{a_0}{2R} w \right\} \tag{6.1}$$

is suggested as a first step of an improvement for the application to cylindrical bodies. The coefficient $-\frac{a_0^2 \rho_0}{2R}$ of the additional term in (6.1) is independent of the parameters M, ν, ω and n. Due to the independence of ν the approximation formula (6.1) can be applied to more arbitrary oscillatory shell deformations of the form

$$w(x,\theta,t) = \cos(n\theta)\, \hat{w}(x)\, e^{i\omega t}, \tag{6.2}$$

where $\hat{w}(x)$ and its first derivative are representable as Fourier

series (provided that for all individual terms the above mentioned inequalities for $|M_1|$ and $|M_2|$ are valid) without a constant term. The last remark reflects the fact that $\nu = 0$ was excluded from our investigations.

For large values of n especially the correction terms of the order $(1/R)^2$ and $(1/R)^3$ in eqs. (4.1a) and (4.6) can become quite significant. For flutter investigations it is often necessary to consider values of n up to the order of 20.

In the following cases a.) $|M_1| < 1$; $|M_2| < 1$; b.) $|M_1| < 1$; $|M_2| > 1$; c.) $|M_1| > 1$; $|M_2| < 1$ (see (5.1)) the linear piston theory expression (4.2) can no longer be considered as a first order approximation for the aerodynamic pressure, as is demonstrated in section 5.

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Table I: The accuracy of the 2-, 3- and 4-term asymptotic expansions

for $\dfrac{H_n^{(1)}(\,|\nu|\,R\sqrt{M^2-1})}{H_n^{(1)'}(\,|\nu|\,R\sqrt{M^2-1})}$, given by eq. (3.4).

$R = 8$ in.; $M = 3$.

n	ν	$\|\nu\|R\ M^2\text{-}1$	Exact value Re	Exact value Im	2-term approxim. Re	2-term approxim. Im	3-term approxim. Re	3-term approxim. Im	4-term approxim. Re	4-term approxim. Im
0	$\frac{\pi}{16}$	4.443	-0.8748	0.4595	-0.1125	-1	-0.1125	-0.9810	-0.1082	-0.9810
10	$\frac{\pi}{16}$		-0.5042	-4.070×10^{-5}	-0.1125	-1	-0.1125	-3.5139	-1.2483	-3.5139
0	$\frac{\pi}{8}$	8.886	-0.0557	-0.9953	-0.0563	-1	-0.0563	-0.9953	-0.0557	-0.9953
10	$\frac{\pi}{8}$		-1.9674	-1.0357	-0.0563	-1	-0.0563	-1.6285	-0.1983	-1.6285
0	$\frac{\pi}{4}$	17.772	-0.0281	-0.9988	-0.0281	-1	-0.0281	-0.9988	-0.0281	-0.9988
10	$\frac{\pi}{4}$		-0.0592	-1.2036	-0.0281	-1	-0.0281	-1.1571	-0.0459	-1.1571
0	$\frac{3\pi}{8}$	26.657	-0.0188	-0.9995	-0.0188	-1	-0.0188	-0.9995	-0.0187	-0.9995
10	$\frac{3\pi}{8}$		-0.0254	-1.0777	-0.0188	-1	-0.0188	-1.0698	-0.0240	-1.0698
0	$\frac{\pi}{2}$	35.543	-0.0141	-0.9998	-0.0141	-1	-0.0141	-0.9997	-0.0141	-0.9997
10	$\frac{\pi}{2}$		-0.0166	-1.0417	-0.0141	-1	-0.0141	-1.0393	-0.0163	-1.0393